Marquette University
2017

COMPETITIVE SCHOLARSHIP EXAMINATION IN
MATHEMATICS

Do not open this booklet until you are directed to do so.

1. Fill out completely the following information about yourself.
   
   PRINT
   Last name  First name  Initial  Phone No.

   ADDRESS
   Street address  City  State  Zip

   Your high school: Name _____________________________ City _____________________________

   High School Counselor or Advisor: _____________________________

2. This examination consists of two parts. The time allowed for each will be approximately 60 minutes. Should you finish Part I early, you may proceed to Part II.

3. Part I consists of 15 objective-type questions. Each question has five possible answers marked: A., B., C., D., E. Only one answer is correct. You are to circle the letter corresponding to the correct response for as many problems as you can.

   Example: If \( x = 5 \) and \( y = -2 \), then \( x + 4y \) is

   \[ A. \quad -3 \quad B. \quad -2 \quad C. \quad -1 \quad D. \quad 0 \quad E. \quad +1. \]

4. Part II consists of 3 subjective-type questions. Show a summary of your work in this booklet for each question you attempt, whether or not you obtain a complete solution. Scratch paper is provided but be sure to show the essential steps of your work concisely in the space provided for each question. Only the work appearing in this booklet will be scored. You will be scored on your method of attack, ingenuity, insight, inventiveness, and logical developments as well as your solutions.

5. Pencils will be provided. No tables, rulers, compasses, protractors, slide rules, calculators, or other aids are permitted.

6. a. The scoring of questions in Part I has been devised to discourage random guessing and will be computed as follows:

   \[ \text{(four times number correct)} - \text{(number wrong)}. \]

   b. The scoring for the three questions in Part II will be 13, 13, and 14 for a total of 40 points. Partial credit will be given so it will be to your advantage to do as much as you are able to do on each question.

7. For the scoring committee. Do not write in the box below.

<table>
<thead>
<tr>
<th>Part I:</th>
<th>Part II:</th>
<th>Score on Part I:</th>
</tr>
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<tbody>
<tr>
<td>No. Correct:</td>
<td>Score on 1:</td>
<td>Score on Part II:</td>
</tr>
<tr>
<td>No. Wrong:</td>
<td>Score on 2:</td>
<td>TOTAL:</td>
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<td>Score on 3:</td>
<td>TOTAL:</td>
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PART I

1. If $\log_2 x = \log_3 x$, then
   \begin{align*}
   (A) & \quad \log_2 3x = \log_3 2x \\
   (B) & \quad \log_2 x = 1 \\
   (C) & \quad 2^x = 3^x \\
   (D) & \quad x = 0 \\
   (E) & \quad \log_2 2x = \log_3 3x
   \end{align*}

2. If $f(x) f\left(\frac{1}{x}\right) = f\left(\frac{1}{x}\right)$ for every nonzero real number $x$, then
   \begin{align*}
   (A) & \quad f(x) = 1 \text{ for some } x \\
   (B) & \quad f(0) = 0 \text{ or } f(0) = 1 \\
   (C) & \quad f(x) = 0 \text{ for some } x \\
   (D) & \quad f(x) = f\left(\frac{1}{x}\right) \text{ for every nonzero real number } x \\
   (E) & \quad f(x_1) = f(x_2) \text{ for all real numbers } x_1 \text{ and } x_2
   \end{align*}

3. Suppose that the positive integer $x_1$ is one of the two distinct solutions $x_1$ and $x_2$ of $x^2 + bx + 1 = 0$. Then
   \begin{align*}
   (A) & \quad -2 \leq b \leq 2 \\
   (B) & \quad b \text{ is a positive fraction} \\
   (C) & \quad x_2 < x_1 \\
   (D) & \quad x_2 > 1 \\
   (E) & \quad -x_1 \text{ and } -x_2 \text{ are solutions of } x^2 + bx - 1 = 0
   \end{align*}

4. If $x + y = 1$ and $x^3 + y^3 = 1$, then
   \begin{align*}
   (A) & \quad x^2 - y^2 = 1 \\
   (B) & \quad x^4 + y^4 = 2 \\
   (C) & \quad x^2 + y^2 = 1 \\
   (D) & \quad xy = 1 \\
   (E) & \quad x = y
5. We use decimal notation: since \( \frac{1111}{101} = 11 \), so \( \frac{1111111111111111}{101} \) equals
   
   (A) 1010101010101
   (B) 11001100110011
   (C) 1001001001001
   (D) 1011011011011
   (E) 11011011011011

6. The graph of \((x + 1)^2 = y^2\)
   
   (A) is a circle
   (B) is a parabola
   (C) is a line
   (D) consists of two perpendicular lines
   (E) is the graph of \(x^2 = (y + 1)^2\)

7. First divide the real number \(x\) by one third and then add five. If the result thus obtained is a positive integer, then
   
   (A) \(x\) is a positive integer
   (B) \(x\) is an integer which is a multiple of 3
   (C) \(x \geq 1\)
   (D) \(1 - 6x\) is an integer
   (E) \(x \leq -\frac{5}{3}\)

8. Let \(f(x) = 1 + x^3 + x^5\) and \(c\) a real number such that \(f(c) = 2\). Then \(\frac{1}{1 + f(-c)}\) equals
   
   (A) -1
   (B) 0
   (C) 1
   (D) \(\frac{1}{2}\)
   (E) 2.
9. A is a point outside a circle with center \( C \) and radius 1. If the lines through \( A \) and tangent to the circle are perpendicular, then \( AC \) equals

(A) 1

(B) 2

(C) \( \sqrt{2} \)

(D) \( \frac{1}{\sqrt{2}} \)

(E) \( \frac{1}{2\sqrt{2}} \)

10. The diagonal \( AC \) of a trapezoid \( ABCD \) bisects both the interior angles at \( A \) and at \( C \). Then this trapezoid is

(A) a square

(B) not a square

(C) a parallelogram

(D) a rectangle

(E) not a rhombus

11. If \( a \) is a positive real number, then \( \sqrt{a - \frac{2}{3}} \) equals

(A) \( \frac{1}{\sqrt{a^{\frac{2}{3}}}} \)

(B) \( a^{\frac{1}{3}} \)

(C) \( -a^{\frac{1}{3}} \)

(D) \( a^{3} \)

(E) \( \frac{1}{a^{\frac{2}{3}}} \)

12. If \( \alpha \) is one of the interior angles of a triangle \( ABC \) and \( \sin \alpha \cos \alpha = -\frac{1}{7} \), then

(A) \( \alpha \) is an acute angle

(B) \( \alpha \) may be an acute angle

(C) \( \alpha = 135^\circ \)

(D) \( \alpha = 150^\circ \) or \( \alpha = 30^\circ \)

(E) the triangle \( ABC \) is a right triangle
13. "x = 9, y = −25" and "x = −25, y = 9" are distinct integer solutions of |xy| = 225. The number of distinct integer solutions of |xy| = 225 is

(A) odd

(B) infinite

(C) a multiple of 4 but not 8

(D) even but not a multiple of 4

(E) a multiple of 8

14. Given that n is an integer such that 1 ≤ n ≤ 10, then the probability that n(n + 1)(n + 2) is a multiple of 9 is

(A) $\frac{1}{10}$

(B) $\frac{1}{5}$

(C) $\frac{2}{5}$

(D) $\frac{1}{2}$

(E) $\frac{3}{10}$

15. The point D is on the hypothenuse BC of the right triangle ABC and AD is perpendicular to BC. If BC = 4 and CD = 1, then AC has length

(A) 2

(B) 4

(C) 3

(D) 1

(E) 5
PART II

1. Let $d$ be a prime number which satisfies the following condition:

   if $r$ is the remainder obtained when dividing 377 by $d$, then $2r$ is the remainder obtained when dividing 421 by $d$.

   Find all possible values of $d$. 

   [13 POINTS]
2. A and B are opposite vertices of a cube whose edges have length 1. The edges of the cube are AA_1, AA_2, AA_3, BB_1, BB_2, BB_3 and the six edges A_iB_j, i \neq j, i, j = 1, 2, 3.  

(a) Give a sketch of this cube and properly label the eight vertices A, A_1, A_2, A_3, B, B_1, B_2, B_3.

(b) The cube is half full of water and is positioned such that the diagonal AB is vertical: the surface of the water takes the shape of a regular hexagon (whose vertices are midpoints of the edges A_iB_j). Find the length of the sides of this hexagon.
3. Arrange the points $A_1, A_2, \ldots, A_{10}$ consecutively and clockwise on the circumference of a circle. A diagram consists of five chords such that

1. each of the ten points is on a chord, and
2. no two chords intersect inside the circle.

\[ \text{14 POINTS} \]

\[ \text{a. Sketch all diagrams for which } A_1 A_6 \text{ is one of the chords.} \]

\[ \text{b. Sketch all diagrams for which } A_1 A_4 \text{ is one of the chords.} \]

\[ \text{c. How many diagrams have exactly 4 chords between consecutive points?} \]